PITT-96-172/ IFP-605-UNC

RELAXATION AND KINETICS IN SCALAR FIELD THEORIES

D. Boyanovsky[(a)], I. D. Lawrie[(b)] and D.-S. Lee[(c)]

(a) Department of Physics and Astronomy, University of Pittsburgh, Pittsburgh, PA. 15260, U.S.A.

(b) Department of Physics and Astronomy, University of Leeds, Leeds LS2 9JT, U.K.

(c) Department of Physics and Astronomy, University of North Carolina Chapel Hill, N.C. 27599, U.S.A.

()

Abstract

A new approach to the dynamics of relaxation and kinetics of thermalization in a scalar field theory is presented that incorporates the relevant time scales through the resummation of hard thermal loops. An alternative derivation of the kinetic equations for the "quasiparticle" distribution functions is obtained that allows a clear understanding of the different "coarse graining" approximations usually involved in a kinetic description. This method leads to a systematic perturbative expansion to obtain the kinetic equations including hard-thermal loop resummation and to an improvement including renormalization, off-shell effects and contributions that change chemical equilibrium on short time scales. As a byproduct of these methods we establish the relation between the relaxation time scale in the linearized equation of motion of the quasiparticles and the thermalization time scale of the quasiparticle distribution function in the "relaxation time approximation". Hard thermal loop resummation dramatically modifies the scattering rate for long wavelength modes as compared to the usual (semi) classical estimate. Relaxation and kinetics are studied both in the unbroken and broken symmetry phases of the theory. The broken symmetry phase also provides the setting to obtain the contribution to the kinetic equations from processes that involve decay of a heavy scalar into light scalar particles in the medium.

11.10.-z; 05.30.-d; 24.60.-k

I. INTRODUCTION

The motivation for understanding non-equilibrium phenomena in intermediate and high energy physics becomes more pressing with the experimental possibility of probing the quark-gluon and chiral phase transitions in upcoming heavy ion colliders at RHIC and large energy-luminosity hadron colliders at LHC. In these high energy, high luminosity experiments ($\sqrt{s} \geq 200$Gev/nucleon) a large energy density ≈ 1Gev/fm^3 is deposited in the collision region corresponding to temperatures $T \approx 200$Mev with time scales for the relevant relaxational processes of order $t \approx 1\text{fm}/c$. In these extreme situations, the challenge is to describe transport and relaxation phenomena from a fundamental microscopic theory since a description of non-equilibrium processes may lead to a better understanding of the experimental signatures of the transition and evolution of the plasma [1]. Any model that attempts to describe the formation and evolution of the quark-gluon plasma and eventual hadronization in heavy ion collisions will have to succeed in providing a description of thermalization and relaxation during the expansion stage. The rate of thermalization and relaxation will determine if an equilibrium thermodynamical description is suitable or a full non-equilibrium treatment will ultimately be necessary to make quantitative statements on the evolution and signatures of the phase transition. Thermalization and relaxation are usually studied via a kinetic approach to non-equilibrium phenomena, typically through Boltzmann equations for the distribution functions. Such a description involves a wide separation of time and length scales and uncontrolled and drastic approximations are often made to simplify the problem. There is considerable effort in obtaining non-equilibrium equations that describe thermalization and relaxation starting from the microscopic theory that describes the full dynamics [2–7]. More recently this program has been extended to a detailed study of the kinetics of QCD and the excitations of non-abelian gauge theories [8–10] and the transport coefficients in scalar theories [11].

Transport equations for quark matter are the subject of intense study since a deeper understanding of the thermalization and hadronic processes needs a full microscopic description. A kinetic description based on QCD has been attempted over a decade ago but remains a formidable formal structure with very difficult implementation. While a description from a fundamental theory is desirable, the technical obstacles are fairly formidable and instead a description in terms of low energy effective theories is emerging [12–14].

A kinetic description of transport and relaxation is justified only when there is a wide separation between the microscopic time and length scales, namely the thermal (or Compton) wavelength (mean separation of particles) and the relaxation scales (mean free path and relaxation time). The usual kinetic approach leading to Boltzmann transport equations involves Wigner transforms of two point (or higher point) correlation functions and eventually a gradient expansion [5,12,13]. Such a gradient expansion assumes that the center of mass Wigner variables are "slowly varying" but it is seldom clear at this level which are the fast and which are the slow scales involved. A "coarse graining" procedure is typically invoked that averages out microscopic scales in the kinetic description [4] leading to irreversible evolution in the resulting equations. Such averaging procedure is usually poorly understood and justified *a posteriori*.

It is recognized that at temperatures much larger than the masses of the particles, it is necessary to perform a resummation of the perturbative series to incorporate the relevant

microscopic time scales in the description of thermal processes. Such a non-perturbative resummation scheme has been proposed by Braaten and Pisarski [15] and since then used to obtain damping rates [10,16]. A program to incorporate self-consistently the microscopic time scales within a kinetic approach has been proposed by Lawrie [17] in a manner that is similar in some respects to the hard-thermal loop resummation. In the quark-gluon plasma this program acquires further significance, since for temperatures $T \approx 200$Mev the thermal masses of the "light quarks" are of order $M_q(T) \approx gT > m_q \approx 5 - 10$Mev with m_q the "current" quark masses and g the quark-gluon coupling constant [9,10,15].

Another realm in which relaxation and thermalization also play a fundamental role is the description of the reheating stage in cosmological inflationary scenarios [18–20]. Recently new non-equilibrium phenomena have been recognized to play a role during the reheating stage in the post-inflationary epoch in early universe cosmology. These phenomena are the result of profuse particle production during the stage of parametric amplification of quantum fluctuations for large amplitudes of the inflaton field [21–24]. A similar mechanism has been recently proposed to play a role in hadronization during the supercooling stage of the quark-gluon phase transition [25]. Since the particles produced during this stage are in a non-equilibrium distribution [23], the final stage of reheating needs a deeper understanding of thermalization processes.

The reliability of any kinetic approach to the description of thermalization and relaxation, either in early universe processes or hadronization and equilibration during the quark-gluon plasma transition, hinges upon a *complete* understanding of the time scales involved. Since in extreme environments the microscopic time scales are modified by the medium, it is important to incorporate these effects in the kinetic description.

The purpose of this article is to provide an alternative program for the description of relaxation and kinetics in scalar field theories. The relevant microscopic and relaxational time scales are incorporated in a consistent first principles calculation starting from the microscopic theory. Such an approach allows us to identify the several different "coarse graining" procedures with a detailed understanding of the relevant scales. The resummation of the hard thermal loops is incorporated consistently in the derivation of the equations of motion that describe relaxation and kinetics and provides a systematic scheme to include off-shell and higher order corrections to the resulting kinetic equations, including terms that provide departure from chemical equilibrium on short time scales, and renormalization effects. We find that the hard thermal loop resummation modifies dramatically the estimate of the scattering rate in the Boltzmann equation for long wavelength modes, from that of a "naive" or (semi)classical argument.

Furthermore, our approach provides a direct proof of the relation between the relaxation processes of quasi-particles and the thermalization rate of the quasiparticle distribution function in the "relaxation time approximation" of the Boltzmann equation that describes their evolution. Such a relation was proposed by Weldon for the identification of the quasi-particle damping rate [26] but to our knowledge a complete proof of such relation has not been given within the context of the Boltzmann equation as derived from a microscopic theory.

Although we focus here on a scalar field theory both in the broken and unbroken symmetry phase to distinguish the different scales and processes, the methods can be extended to other models and theories almost straightforwardly (obviously with the complications of

gauge invariance in gauge theories). The broken symmetry case allows us to study the situation in which the scalar field is coupled to other lighter scalars with a trilinear coupling and to include decay and recombination processes in both the relaxation and kinetic description.

In the next section we introduce the model and the techniques. In section III we study the relaxation of "quasi-particles" obtaining the real-time equation of motion in the linearized approximation, including consistently the hard-thermal-loop corrections to the self-energy. In section IV we introduce a new formulation of kinetic theory that incorporates the hard-thermal-loop corrections consistently, displays all the relevant time scales and leads to a well defined "coarse-graining" procedure. This formulation allows a systematic improvement on the kinetic description and, if necessary, consistently includes off-shell effects and renormalization in a computationally accesible form.

Section V presents our conclusions, summarizes the main results and poses new questions.

II. THE MODEL AND TECHNIQUES

We will restrict our study to a self-interacting scalar theory both in the broken and unbroken symmetry state. The Lagrangian density is given by:

$$\mathcal{L}_0 = \frac{1}{2}\left(\partial_\mu \Phi\right)^2 - \frac{1}{2} m_0^2 \Phi^2 - \frac{\lambda_0}{4!}\Phi^4 \; . \tag{2.1}$$

with m_0 ; λ_0 the bare parameters.

As mentioned in the introduction, the first step towards understanding the kinetic regime is the identification of the *microscopic* time scales in the problem. In a medium, the bare particles are dressed by the interactions becoming "quasiparticles". One is interested in describing the relaxation of these quasiparticles. Thus the important microscopic time scales are those associated with the quasiparticles and not the bare particles. If a Boltzmann equation is obtained in some perturbative scheme, such a scheme should be in terms of the quasiparticles, which already implies a resummation of the perturbative expansion. This is precisely the rationale behind the resummation of the hard thermal loops in finite temperature field theory [15] and also behind the self-consistent treatment proposed by Lawrie [17]. In this scalar theory such a resummation can be implemented by writing in the Lagrangian

$$m_0^2 = m_R^2(T) + \delta m^2(T) \tag{2.2}$$

where $m_R^2(T)$ is the renormalized and *temperature dependent* quasiparticle mass which enters in the propagators and $\delta m^2(T)$ is a counterterm which will cancel a subset of Feynman diagrams in the perturbative expansion and is considered part of the interaction Lagrangian. This method is the simplest form of implementing the hard thermal loop resummation in the scalar case and was used within this context by Parwani [27]. We note that this renormalized, temperature dependent mass determines the important time scales in the medium but is *not* the position of the quasiparticle pole (or, stricly speaking, resonance: see the discussion below). We could also introduce counterterms for wave-function and coupling constant renormalization and proceed to a perturbative expansion of the BPHZ type in terms of the renormalized couplings and fields. Such a possibility will be discussed later within the context of improvements on the method and renormalization.

A. Non- Equilibrium Techniques

The field theoretical methods to describe processes out of equilibrium are known and described at length in the literature [28–32]. The basic ingredient is the time evolution of an initially prepared density matrix, which leads to the generating functional of non-equilibrium Green's functions in terms of a path integral representation along a contour in the complex time plane. This contour involves a forward time branch, a backward time branch and a third branch down the imaginary time axis to time $\tau = -i\beta$ if the initial density matrix describes an equilibrium ensemble at initial temperature $1/\beta$. For the computation of real-time correlation functions at the order of approximation we consider, the only role of the imaginary time branch is to determine the boundary conditions on the propagators.

The fields living on the forward and backward branches will be labelled with + and −, respectively, and the effective Lagrangian that enters in the path integral representation of the non-equilibrium generating functional is given by

$$\mathcal{L}_{\rm noneq} = \mathcal{L}[\Phi^+] - \mathcal{L}[\Phi^-]. \tag{2.3}$$

From this path integral representation it is possible to construct a perturbative expansion of the non-equilibrium Green's functions in terms of modified Feynman rules. Correlation functions are obtained as functional derivatives with respect to sources $j^\pm$ on the respective branches. The non-equilibrium Feynman rules are:

i) The number of vertices is doubled: those in which all the fields are on the + branch are the usual interaction vertices, while those in which the fields are on the − branch have the opposite sign.

ii) The combinatoric factors are the same as in usual field theory.

iii) The spatial Fourier transform of the (bosonic) propagators are

$$G_k^{++}(t,t') = G_k^{>}(t,t')\Theta(t-t') + G_k^{<}(t,t')\Theta(t'-t)\ , \tag{2.4}$$
$$G_k^{--}(t,t') = G_k^{>}(t,t')\Theta(t'-t) + G_k^{<}(t,t')\Theta(t-t')\ , \tag{2.5}$$
$$G_k^{+-}(t,t') = -G_k^{<}(t,t')\ , \tag{2.6}$$
$$G_k^{-+}(t,t') = -G_k^{>}(t,t')\ , \tag{2.7}$$
$$G_k^{>}(t,t') = i\int d^3x\ e^{-i\vec{k}\cdot\vec{x}}\ \langle\Phi(\vec{x},t)\Phi(\vec{0},t')\rangle\ , \tag{2.8}$$
$$G_k^{<}(t,t') = i\int d^3x\ e^{-i\vec{k}\cdot\vec{x}}\ \langle\Phi(\vec{0},t')\Phi(\vec{x},t)\rangle\ . \tag{2.9}$$

where Φ is the bose field. Now we have to specify the properties of the initial state. Particularly convenient is the choice of a thermal initial state with temperature T. Then the density matrix of this initial state is $\rho = e^{-H_0/T}$, where H_0 is the Hamiltonian for times $t < 0$. This choice of the initial state determines the boundary conditions on the Green's functions. These are the usual periodicity conditions in imaginary time (KMS conditions):

$$G^{<}(\vec{x},t;\vec{x}',t') = G^{>}(\vec{x},t-i\beta;\vec{x}',t') \tag{2.10}$$

Finally, the free-field Green's functions for bose fields are constructed from the following ingredients:

$$G_k^{>}(t,t') = \frac{i}{2\omega_k}\left\{[1+n(\omega_k)]e^{-i\omega_k(t-t')} + n(\omega_k)e^{i\omega_k(t-t')}\right\} \ , \tag{2.11}$$

$$G_k^{<}(t,t') = \frac{i}{2\omega_k}\left\{[1+n(\omega_k)]e^{i\omega_k(t-t')} + n(\omega_k)e^{-i\omega_k(t-t')}\right\} \ , \tag{2.12}$$

$$\omega_k = \sqrt{\vec{k}^2 + m_R^2(T)} \ , \qquad n_b(\omega_k) = \frac{1}{e^{\beta\omega_k}-1} \ , \tag{2.13}$$

An important property that will be used in the calculations that follow is the relation

$$G_k^{>}(t,t') = G_k^{<}(t',t) \tag{2.14}$$

III. LINEAR RELAXATION

In this section we study the relaxation of quasiparticles, from the linearized real-time equation of motion for the *expectation value* of the scalar field

$$\phi(\vec{x},t) \equiv \langle \Phi^+(\vec{x},t)\rangle = \langle \Phi^-(\vec{x},t)\rangle \tag{3.1}$$

where $\Phi^{\pm}$ are the fields on the forward and backward branches of the contour respectively and the expectation value is in the ensemble of the initial density matrix. We derive the equation of motion for $\phi(\vec{x},t)$ using the tadpole method [33,34], which consists in splitting the field Φ into its expectation value and a fluctuation as

$$\Phi^{\pm}(\vec{x},t) = \phi(\vec{x},t) + \psi^{\pm}(\vec{x},t) \ , \tag{3.2}$$

where $\psi^{\pm}$ are the fluctuation operators defined along the respective branches. The effective equation of motion for the field $\phi(\vec{x},t)$ follows from the condition:

$$< \psi^{\pm}(\vec{x},t) >= 0 \ . \tag{3.3}$$

This tadpole method is different from that used by Elmfors *et al* [35] in that the present method is implemented with the non-equilibrium, real time functional and Green's functions.

After the shift (3.2), the non-equilibrium Lagrangian density reads

$$\mathcal{L}[\phi+\psi^+] - \mathcal{L}[\phi+\psi^-] = \left\{\frac{\delta\mathcal{L}}{\delta\phi}\psi^+ + \mathcal{L}_0[\psi^+] - \lambda\left(\frac{\phi^2(\psi^+)^2}{4} + \frac{\phi(\psi^+)^3}{6} + \frac{(\psi^+)^4}{4!}\right)\right.$$
$$\left. -\frac{1}{2}\delta m^2(T)(2\phi\psi^+ + (\psi^+)^2)\right\} - \left\{\psi^+ \to \psi^-\right\} \tag{3.4}$$

with $\mathcal{L}_0[\psi^{\pm}]$ the free field Lagrangian density of a field with mass $m_R(T)$. To implement the tadpole method one considers the *linear*, cubic and quartic terms and counterterm as perturbations and the condition (3.3) is imposed order by order in a perturbative expansion, using the Feynman rules given above [33,34].

We now analyze different cases in turn.

A. $T >> T_c$: Unbroken symmetry case

We are interested in studying relaxation and kinetics both in the disordered high temperature phase as well as in the ordered low temperature phase. This is achieved by allowing the zero temperature renormalized squared mass to be negative, that is

$$m_R^2(T=0) = -|m_R^2(T=0)|. \tag{3.5}$$

When the temperature is much larger than the renormalized (zero temperature) mass, the hard-thermal loop resummation is needed to incorporate the physically relevant time and length scales in the perturbative expansion. In the equation of motion this is achieved by requiring that the mass counterterm $\delta m^2(T)$ cancel the 1-loop tadpole contribution.

This leads to the following self-consistent "gap" equation for $m_R^2(T)$ [27,36]:

$$m_R^2(T) = -m_0^2 + \frac{\lambda}{2}\int \frac{dk}{2\pi^2}\frac{k^2}{2\sqrt{k^2+m_R^2(T)}}\left\{1+\frac{2}{e^{\beta\sqrt{k^2+m_R^2(T)}}-1}\right\} \tag{3.6}$$

The divergence in the zero temperature part of (3.6) (quadratic and logarithmic in terms of a spatial momentum cutoff) can be absorbed in a renormalization of the bare mass by a subtraction at some renormalization scale:

$$m_R^2(T) = -|m_R^2| + \frac{\lambda}{2}\int \frac{dk}{2\pi^2}\frac{k^2}{2\sqrt{k^2+m_R^2(T)}}\left\{1+\frac{2}{e^{\beta\sqrt{k^2+m_R^2(T)}}-1}\right\} - \text{subtraction} \tag{3.7}$$

where both m_R^2 and the subtraction depend on the renormalization scale and the renormalization prescription. Now the gap equation is finite. For $T \gg m_R(T)$ we obtain

$$\begin{aligned} m_R^2(T) &= -|m_R^2| + \frac{\lambda}{2}\left\{\frac{T^2}{12} - \frac{m_R(T)T}{4\pi} + \mathcal{O}\left(m_R(T)\log\frac{m_R^2(T)}{T^2}\right)\right\} \qquad \text{for} \quad T > m_R(T) \\ &= \frac{\lambda T^2}{24} + \mathcal{O}\left(\lambda^{\frac{3}{2}}T^2\right) \qquad \text{for} \quad T > \sqrt{\lambda}T > |m_R| \end{aligned} \tag{3.8}$$

The leading term of the last expression of (3.8) provides the correct microscopic time scale. In the massless case it serves as an infrared cutoff for the loop integrals [27,35]. We will concentrate on the *linear relaxation* case in which the equations of motion for the expectation value are linearized in the amplitude of $\phi(\vec{x},t)$. This approximation is valid for small amplitude fluctuations from the minimum of the finite temperature effective potential in the unbroken symmetry phase. Two-loop diagrams contributing to the equation of motion obtained from the condition $\langle\psi^+\rangle = 0$ are shown in figure 1. This condition leads to

$$\ddot{\phi}(\vec{x}',t') - \nabla^2\phi(\vec{x}',t') + m_R^2(T)\phi(\vec{x}',t') + \int d^3x''dt''\Sigma_{ret}(\vec{x'}-\vec{x''},t'-t'')\phi(\vec{x}'',t'') = 0 \tag{3.9}$$

The mass counterterm has been used to cancel the 1-loop tadpoles. To two-loop order we find

$$\Sigma_{ret}(\vec{x}-\vec{x'},t-t') = -i\frac{\lambda^2}{6}[(-iG^>(\vec{x}-\vec{x'},t-t'))^3 - (-iG^<(\vec{x}-\vec{x'},t-t'))^3]\Theta(t-t') \tag{3.10}$$

The equation obtained from $\langle \psi^- \rangle = 0$ is the same as (3.9) as a consequence of unitarity.

Translational invariance of the self-energy makes it convenient to write the equation of motion for the spatial Fourier transform of ϕ. We therefore introduce

$$\phi(\vec{x},t) = \int \frac{d^3p}{(2\pi)^3}\, e^{i\vec{p}\cdot\vec{x}}\, \phi_{\vec{p}}(t) \tag{3.11}$$

$$\Sigma_{ret}(\vec{x}-\vec{x},t-t') = \int \frac{d^3p}{(2\pi)^3}\, e^{i\vec{p}\cdot(\vec{x}-\vec{x}')}\, \Sigma_{\vec{p}}(t-t') \tag{3.12}$$

In order to solve the integro-differential equation, we will impose the initial condition $\dot{\phi}_{\vec{p}}(t<0) = 0$ such that this configuration is "released" at time $t = 0$ [33]. Under this condition the evolution equation becomes

$$\ddot{\phi}_{\vec{p}}(t) + \omega_p^2 \phi_{\vec{p}}(t) + \int_0^t dt' \Sigma_{\vec{p}}(t-t')\phi_{\vec{p}}(t') = 0 \tag{3.13}$$

where

$$\omega_p^2 = |\vec{p}|^2 + m_R^2(T) \tag{3.14}$$

$$\begin{aligned}\Sigma_{\vec{p}}(t-t') = -\frac{\lambda^2}{3}\int \frac{d^3k_1}{(2\pi)^3}\,\frac{d^3k_2}{(2\pi)^3}\,\frac{d^3k_3}{(2\pi)^3}\,\frac{1}{8\omega_{k_1}\omega_{k_2}\omega_{k_3}}\left[(2\pi)^3\delta^3(\vec{p}-\vec{k}_1-\vec{k}_2-\vec{k}_3)\right] \\ \{\sin\left[(\omega_{k_1}+\omega_{k_2}+\omega_{k_3})(t-t')\right]\left[(1+n_{k_1})\,(1+n_{k_2})\,(1+n_{k_3}) - \; n_{k_1}\,n_{k_2}\,n_{k_3}\,\right] \\ + 3\sin\left[(\omega_{k_1}+\omega_{k_2}-\omega_{k_3})(t-t')\right]\left[(1+n_{k_1})\,(1+n_{k_2})\,n_{k_3} - \; n_{k_1}\,n_{k_2}\,(1+n_{k_3})\right]\}\end{aligned} \tag{3.15}$$

The linearized equation of motion (3.13) can now be solved by the Laplace transform. With the boundary conditions $\phi_{\vec{p}}(t=0) = \phi_{i;\vec{p}}\,; \dot{\phi}_{\vec{p}}(t=0) = 0$ and denoting the Laplace transforms of $\phi_{\vec{p}}(t)\,, \Sigma_{\vec{p}}(t)$ by $\phi_{\vec{p}}(s)\,, \Sigma_{\vec{p}}(s)$ respectively (here s is the Laplace transform variable) we find

$$\phi_{\vec{p}}(s) = \frac{\phi_{i;\vec{p}}\; s}{s^2 + \omega_p^2 + \Sigma_{\vec{p}}(s)} \tag{3.16}$$

The time evolution is found by inverse Laplace transform, i.e. by integration in the complex s-plane along the Bromwich contour

$$\phi_{\vec{p}}(t) = \int_{-i\infty+\epsilon}^{i\infty+\epsilon} e^{st}\phi_{\vec{p}}(s)\frac{ds}{2\pi i} \tag{3.17}$$

with ϵ to the right of the real part of all the singularities of the Laplace transform $\phi_{\vec{p}}(s)$. The Laplace transform of the retarded self-energy has the following spectral representation

$$\Sigma_{\vec{p}}(s) = -\int \frac{2\,p_0\,\rho(p_0,\vec{p};T)}{s^2+p_0^2} dp_o\,. \tag{3.18}$$

where $\rho(p_0,\vec{p};T)$ is the spectral density, given to two- loop order by

$$\rho(\vec{k},p_0;T) = \frac{\lambda^2}{6}\int \frac{d^3k_1}{(2\pi)^3}\,\frac{d^3k_2}{(2\pi)^3}\,\frac{d^3k_3}{(2\pi)^3}\,\frac{1}{8\omega_1\omega_2\omega_3}\,\left[(2\pi)^3\delta^3(\vec{k}-\vec{k}_1-\vec{k}_2-\vec{k}_3)\right]$$
$$\{\delta(p_0-\omega_{k1}-\omega_{k2}-\omega_{k3})\,((1+n_{k1})\,(1+n_{k2})\,(1+n_{k3})-n_{k1}\,n_{k2}\,n_{k3})$$
$$+3\,\delta(p_0+\omega_{k1}-\omega_{k2}-\omega_{k3})\,(n_{k1}\,(1+n_{k2})\,(1+n_{k3})-(1+n_{k1})\,n_{k2}\,n_{k3})\} \tag{3.19}$$

This spectral density displays two different type of contributions [26,16,37]: the first one (corresponding to the first δ-function) corresponds to the production process $\phi \to \phi\phi\phi$ with the Boltzmann weight $(1+n_{k1})(1+n_{k2})(1+n_{k3})$ minus the recombination $\phi\phi\phi \to \phi$ in the medium with the weight $n_{k1}n_{k2}n_{k3}$. The second contribution (second δ-function) corresponds to the scattering process $\phi\phi \to \phi\phi$ with weight $n_{k1}(1+n_{k2})(1+n_{k3})$ minus the inverse process with weight $n_{k1}n_{k2}(1+n_{k3})$. The self-energy $\Sigma_{\vec{p}}(s)$ has an imaginary part along the imaginary axis in the s-plane which can be read off from the spectral representation:

$$\Sigma_{I,\vec{p}}(i\omega \pm 0^+) = \pm\pi\ \mathrm{sign}\,(\omega)\,[\rho(|\omega|,\vec{p},T)-\rho(-|\omega|,\vec{p},T)] \tag{3.20}$$

where the sign(ω) reflects the retarded nature [32,37]. To two-loop order we find the imaginary part to be given by:

$$\Sigma_{I,\vec{k}}(i\omega+0^+) = \tfrac{\lambda^2}{6}\,\pi\mathrm{sign}(\omega)\int \frac{d^3k_1}{(2\pi)^3}\,\frac{d^3k_2}{(2\pi)^3}\,\frac{d^3k_3}{(2\pi)^3}\,\frac{1}{8\omega_1\omega_2\omega_3}\,\left[(2\pi)^3\delta^3(\vec{k}-\vec{k}_1-\vec{k}_2-\vec{k}_3)\right]$$
$$\times[\delta(|\omega|-\omega_{k1}-\omega_{k2}-\omega_{k3})\mathcal{N}_1$$
$$+3\delta(|\omega|+\omega_{k1}-\omega_{k2}-\omega_{k3})\mathcal{N}_2$$
$$+\delta(-|\omega|-\omega_{k1}-\omega_{k2}-\omega_{k3})\mathcal{N}_3$$
$$+\delta(-|\omega|+\omega_{k1}-\omega_{k2}-\omega_{k3})\mathcal{N}_4] \tag{3.21}$$

where

$$\mathcal{N}_1 = (1+n_{k1})\,(1+n_{k2})\,(1+n_{k3}) - n_{k1}\,n_{k2}\,n_{k3}$$
$$\mathcal{N}_2 = n_{k1}\,(1+n_{k2})\,(1+n_{k3}) - (1+n_{k1})\,n_{k2}\,n_{k3}$$
$$\mathcal{N}_3 = n_{k1}\,n_{k2}\,n_{k3} - (1+n_{k1})\,(1+n_{k2})\,(1+n_{k3})$$
$$\mathcal{N}_4 = (1+n_{k1})\,n_{k2}\,n_{k3} - n_{k1}\,(1+n_{k2})\,(1+n_{k3}) \tag{3.22}$$

This result coincides with that found by Wang and Heinz [37]. The first term gives a contribution above the three particle threshold at $\omega > 3m_R(T)$, the third term vanishes identically, the second and fourth term give the collisional contribution and have support all along the imaginary axis [37]. Therefore along the imaginary axis we find

$$\Sigma_{\vec{k}}(i\omega \pm 0^+) = \Sigma_{R,\vec{k}}(\omega) \pm i\Sigma_{I,\vec{k}}(\omega) \tag{3.23}$$

The real part can be obtained by a dispersion relation. It is very difficult to obtain in general, but we find that it is quadratically and logarithimically divergent. These divergences can be subtracted away from the dispersion integral and the subtraction absorbed in a mass and wave function renormalization.

In order to invert the Laplace transform and obtain the real-time evolution, we need to understand the analytic structure of $\phi_{\vec{p}}(s)$ in the complex s-plane. Isolated single particle poles would be at the values $s = \pm i\Omega(\vec{p},T)$ with $\Omega(\vec{p},T)$ the solutions of

$$-\Omega^2(\vec{p},T)+|\vec{p}|^2+M_\sigma^2(T)+\Sigma(s=\pm i\Omega,\vec{p},T)=0 \tag{3.24}$$

Since the self-energy is complex along the imaginary axis, such a pole solution would be complex. However it is easy to see that there is no solution to (3.24) in the first (physical) Riemann sheet. The pole moves off into the second (unphysical) Riemann sheet as corresponds to a "resonance" (or quasiparticle pole) [33]. The discontinuity of the Laplace transform across the imaginary axis is given by:

$$\phi(\vec{p},s=i\omega+0^+)-\phi(\vec{p},s=i\omega-0^+)=\frac{\phi_{i;\vec{p}}\;\;2i\omega\,\Sigma_I(\omega,\vec{p},T)}{[\omega^2-|\vec{p}|^2-M_\sigma^2-\Sigma_R(\omega,\vec{p},T)]^2+\Sigma_I(\omega,\vec{p},T)^2}. \tag{3.25}$$

This discontinuity is the only singularity in the (first Riemann sheet) complex s-plane and it vanishes at the origin. This fact allows us to deform the integration countour to wrap around the cut.

Finally for the time evolution we obtain

$$\phi_{\vec{p}}(t)=\frac{2\phi_{i;\vec{p}}}{\pi}\int_0^\infty\frac{\omega\Sigma_I(\omega,\vec{p};T)\cos(\omega t)\,d\omega}{[\omega^2-\omega_p^2-\Sigma_R(\omega,\vec{p};T)]^2+\Sigma_I(\omega,\vec{p};T)^2} \tag{3.26}$$

with ω_p^2 given by (3.14).

Setting $t=0$ in the above expression leads to the sum rule

$$\frac{2}{\pi}\int_0^\infty\frac{\omega\Sigma_I(\omega,\vec{p};T)\,d\omega}{[\omega^2-\omega_p^2-\Sigma_R(\omega,\vec{p};T)]^2+\Sigma_I^2(\omega,\vec{p};T)}=1 \tag{3.27}$$

which has been previously obtained by Pisarski [38] in a rather different manner.

For very weak coupling the integrand in (3.26) features a narrow resonance at $\omega^2=\omega_p^2+\mathcal{O}(\lambda^2)$, and can be approximated by a Breit-Wigner form leading to the real-time evolution [33]

$$\phi_{\vec{p}}(t)=\phi_{i;\vec{p}}Z[\vec{p};T]e^{-\Gamma(\vec{p};T)t}\cos(\omega_p t+\alpha) \tag{3.28}$$

with

$$\Gamma(\vec{p};T)\approx Z[\vec{p},T]\frac{\Sigma_I(\omega=\omega_p,\vec{p};T)}{2\omega_p} \tag{3.29}$$

$$Z[\vec{p};T]=\left[1-\frac{\partial\Sigma_R(\omega,\vec{p},T)}{\partial\omega^2}\bigg|_{\omega=\omega_p}\right]^{-1}=1+\mathcal{O}(\lambda^2) \tag{3.30}$$

$$\alpha=-\frac{\partial\Sigma_I(\omega,\vec{p},T)}{\partial\omega^2}\bigg|_{\omega=\omega_p} \tag{3.31}$$

where $Z[\vec{p};T]$ is the wave function renormalization defined on shell. To the order that we are working, this will be set to one in (3.29). $\Gamma(\vec{p};T)$ is the *collisional* relaxation rate. The Breit-Wigner approximation, however, is valid only for times $t\leq\frac{1}{\Gamma}\ln\left(\frac{\omega_p}{\Gamma}\right)$. For longer times the fall-off is with a power law determined by the behavior of the spectral density at small frequency [33].

The calculation of the decay rate $\Gamma(\vec{p};T)$ is in general very difficult for arbitrary momentum and temperature (see however [37]). For $p \ll M_R(T) \ll T$ we can approximate it by the zero momentum limit, which can be calculated relatively easily. The only contribution to the imaginary part on-shell is given by the second and fourth terms (collisional terms) in (3.21).

We find for $T \gg \sqrt{\lambda}T \gg |m_R|$:

$$\Gamma(\vec{p}=0;T) = \frac{\lambda^{3/2}T}{64\sqrt{24\pi}} \tag{3.32}$$

The same result had been obtained previously by many authors [27,35,37]

B. $T << T_c$: Broken symmetry case

In this section we consider the linear relaxation of fluctuations around the broken symmetry state, for non-zero temperature but below the critical value. In this case the scalar field acquires an expectation value v which at tree level is given by

$$v = \sqrt{\frac{6|m_0^2|}{\lambda}} \tag{3.33}$$

The small amplitude fluctuations around the broken symmetry state have a tree level mass squared given by

$$M_0^2 = m_0^2 - \frac{\lambda v^2}{2} = 2|m_0^2| \tag{3.34}$$

We now write the field $\Phi^{\pm}$ as

$$\Phi^{\pm}(\vec{x},t) = v + \phi(\vec{x},t) + \psi^{\pm}(\vec{x},t)\ . \tag{3.35}$$

We will *not* fix v at the tree level value given by eq. 3.33, but instead the v.e.v will be determined order by order in the perturbative expansion by requiring that the linearized equation of motion for $\phi(\vec{x},t)$ be homogeneous. Furthermore as in the previous case we will write $M_0^2 = M_R^2(T) + \delta m^2(T)$ and treat $\delta m^2(T)$ as a counterterm. The microscopic time scale is determined by $M_R^{-1}(T)$. The condition $\langle \psi^{\pm}(\vec{x},t)\rangle = 0$ implemented order by order leads to the equation of motion for $\phi(\vec{x},t)$ as before [33]. In the broken symmetry phase there are new Feynman diagrams that contribute an absorptive part to the self-energy at *one-loop* order which are proportional to the v.e.v. Now there are new vertices in the "shifted" Lagrangian density. Those that contribute to the linearized equation of motion are obtained from the part of the Lagrangian density given by

$$\mathcal{L}_v = \left\{ v\psi^+(m_0^2 - \frac{\lambda}{6}v^2) - \frac{\lambda v}{6}(3\phi(\psi^+)^2 + (\psi^+)^3) \right\} - \left\{ \psi^+ \to \psi^- \right\} \tag{3.36}$$

To one-loop order there are two tadpole terms that contribute to the equation of motion. One is independent of ϕ and is absorbed in a renormalization of v (since it gives an inhomogeneous contribution to the equation of motion). The other is proportional to ϕ and

is cancelled by the mass counterterm leading to the "gap equation" (3.6, 3.7), but in the present case $M_R(T) > T \gg \sqrt{\lambda}T$. The remaining one-loop diagrams contributing to the self-energy with an absorptive part are shown at the top of figure 2. We obtain the equation of motion given by eq. (3.9) with a new one-loop contribution to the retarded self-energy as well as the two-loop contribution obtained in the previous section and given by eq. (3.10). We find the one loop contribution to be given by (see also [33,39])

$$\Sigma^v_{ret}(\vec{x}-\vec{x'},t-t') = -i\frac{\lambda^2 v^2}{2}[(-iG^>(\vec{x}-\vec{x'},t-t'))^2 - (-iG^<(\vec{x}-\vec{x'},t-t'))^2]\Theta(t-t') \tag{3.37}$$

with the following spatial Fourier transform:

$$\Sigma^v_{\vec{p}}(t-t') = -\frac{\lambda^2 v^2}{2}\int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k\omega_{\vec{k}+\vec{p}}}\{(1+2n_k)\sin[(\omega_{\vec{k}+\vec{p}}+\omega_k)(t-t')] \\ +2n_k\sin[(\omega_{\vec{k}+\vec{p}}-\omega_k)(t-t')]\} \tag{3.38}$$

The Laplace transform can again be written in a spectral representation as given by (3.18), and we find the one-loop contribution to the spectral density to be

$$\rho^v(\vec{p},p_o) = \frac{\lambda^2 v^2}{4}\int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}}\omega_{\vec{k}+\vec{p}}}\{(1+2n_k)\delta(p_o-\omega_{\vec{k}}-\omega_{\vec{k}+\vec{p}}) - 2n_k\delta(p_o-\omega_{\vec{k}}+\omega_{\vec{k}+\vec{p}})\} \tag{3.39}$$

The imaginary part is obtained from (3.20), and after analyzing the kinematical regions we find for $\omega > 0$ (for $\omega < 0$ there is a sign change) (see also [39])

$$Im\Sigma^v_{\vec{p}}(i\omega+0^+,T) = \left\{\frac{\lambda^2 v^2}{32\pi}\sqrt{1-\frac{4M_R^2(T)}{\omega^2-p^2}} + \frac{\lambda^2 v^2 T}{16\pi p}\ln\left[\frac{1-e^{-\beta\omega_p^+}}{1-e^{-\beta\omega_p^-}}\right]\right\}\Theta(\omega^2-p^2-4M_R^2(T)) \\ +\frac{\lambda^2 v^2 T}{16\pi p}\ln\left[\frac{1-e^{-\beta\omega_p^+}}{1-e^{-\beta\omega_p^-}}\right]\Theta(p^2-\omega^2) \tag{3.40}$$

$$\omega_p^\pm = \left|\frac{\omega}{2}\pm\frac{p}{2}\sqrt{1-\frac{4M_R^2(T)}{\omega^2-p^2}}\right| . \tag{3.41}$$

The processes that contribute to the first term above are the creation of two mesons in the medium with Boltzmann weight $(1+n_{\vec{k}})(1+n_{\vec{k}+\vec{p}})$ minus the inverse process with weight $n_{\vec{k}}n_{\vec{k}+\vec{p}}$, whereas the processes that contribute to the second term are scattering off quanta in the bath with weight $(1+n_{\vec{k}})n_{\vec{k}+\vec{p}}$ minus its inverse with weight $(1+n_{\vec{k}+\vec{p}})n_{\vec{k}}$ [26,16]. Notice that whereas the first term displays the usual two particle threshold, the second term is reminiscent of Landau damping in that it only has support below the light cone, for $|p| > |\omega|$ [10,16]. Neither term contributes on-shell. However there is a new contribution of $\mathcal{O}(\lambda)$ to mass renormalization and a finite contribution of the same order to wave function renormalization. In terms of the renormalized mass, the position of the resonance obtains a correction of $\mathcal{O}(\lambda)$ but to this order the width remains the same since the imaginary part on shell is already of $\mathcal{O}(\lambda^2)$.

The results of this section also apply if there is a *trilinear* coupling of the scalar field Φ to lighter scalars of the form $g\Phi\chi^2$. In this case the result (3.40) applies (with $\lambda^2 v^2 \to 4g^2$) but with the mass $M_R(T)$ in the argument of the square roots in equations (3.40, 3.41) replaced by $m_\chi(T)$ the (thermal) mass of the χ particle in the loop. If $M_R(T) > 2m_\chi(T)$, decay is kinematically allowed (on-shell) and the scalar particle will acquire a contribution from the imaginary part of the self-energy on shell from the decay process $\Phi \to \chi\chi$ minus the inverse (recombination) process $\chi\chi \to \Phi$. The decay (minus recombination) width for weak coupling is therefore(see also [39])

$$\Gamma_{\Phi\leftrightarrow\chi\chi} \approx \frac{g^2}{16\pi\omega_p}\sqrt{1-\frac{4m^2_\chi(T)}{M^2_R(T)}} + \frac{g^2 T}{8\pi p\omega_p}\ln\left[\frac{1-e^{-\beta W^+_p}}{1-e^{-\beta W^-_p}}\right] \tag{3.42}$$

$$W^\pm_p = \left|\frac{\omega_p}{2} \pm \frac{p}{2}\sqrt{1-\frac{4m^2_\chi(T)}{M^2_R(T)}}\right| \ . \tag{3.43}$$

Now the collisional and the decay width add up. The Landau damping term does not contribute to the width of the particle *directly*, because it has support only below the light cone. Notice, however, that the Landau damping contribution to (3.42) is the same as the finite temperature contribution above the two particle threshold in the first term of (3.40). The sum rule (3.27) ensures that the Landau damping contribution is "borrowed" from the spectral density above the two particle threshold and thus contributes *indirectly* to the width of the particle and to the wave function renormalization (the weight of the quasiparticle pole).

The main purpose of studying the broken symmetry phase is to compare to the kinetic description and recognize the terms corresponding to decay and recombination processes in the evolution equation for the distribution function to be studied in the next section.

IV. KINETIC THEORY

In this section we obtain the evolution equations for the distribution functions of quasiparticles, including off-shell effects. Here we present an alternative to the usual derivation in which correlation functions are written in terms of relative and "center of mass" spacetime coordinates and the Wigner transform in the relative coordinates is performed. In this approach the Boltzmann equation is obtained in a gradient expansion assuming that the dependence on "center of mass" coordinates is weak. We will not assume such a situation, but instead analyze which are the relevant time scales over which a coarse graining procedure must be implemented.

Let us begin by writing the Hamiltonian of the theory as

$$H = H_0 + H_{int} \tag{4.1}$$

$$H_0 = \frac{1}{2}\int d^3x \left\{\Pi^2 + (\nabla\Phi)^2 + M^2_R(T)\Phi^2\right\} \ , \tag{4.2}$$

Where the mass counterterm has been absorbed in the interaction. The interaction part H_{int} will depend on whether $T >> T_c$ or $T < T_c$ is considered. The part of the Hamiltonian H_0 describes free quasiparticles of renormalized finite temperature mass $M_R(T)$ and

is diagonalized in terms of creation and annihilation of free quasiparticle operators $a_k^\dagger$; a_k respectively. The pole mass of the quasiparticles will acquire corrections in perturbation theory, but these will remain perturbatively small at large temperatures.

With this definition, the lifetime and weight of the quasiparticles will be a consequence of interactions. In this manner, the hard thermal loops which in this theory amount to local terms, have all been absorbed in the definition of the thermal mass, which guarantees that the microscopic time scales are explicit in the quasiparticle hamiltonian. Such an approach has been previously advocated by Lawrie [17].

Now consider the case in which at an initial time (to be taken as $t = t_0$) the density matrix is diagonal in the occupation number of the free quasiparticles, but out of equilibrium, with initial occupation numbers $N_k(t_0)$. A comment is in order here. In an initial situation out of equilibrium the temperature is not a meaningful quantity and conceptually $M_R(T)$ in the above Hamiltonian H_0 is not well defined. However we are interested in small departures from equilibrium as is implicit in a kinetic description, thus the value of the temperature in the resummed mass term should be thought of as the final equilibrium temperature. In particular for the relaxation time approximation a typical situation considered is that only few modes are out of equilibrium whereas all other are in equilibrium and serve as the "bath". In this case the temperature T in the mass parameter is that of the modes in equilibrium. This will become clear in the relaxation time approximation discussed later.

The Heisenberg field operators at time t are now written as

$$\Phi(\vec{x},t) = \frac{1}{\sqrt{\Omega}}\sum_k \Phi_k(t)\, e^{i\vec{k}\cdot\vec{x}}\;, \qquad \Phi_k(t) = \frac{1}{\sqrt{2\omega_k}}\Big(a_k(t) + a^\dagger_{-k}(t)\Big)$$
$$\Pi(\vec{x},t) = \frac{1}{\sqrt{\Omega}}\sum_k \Pi_k(t)\, e^{i\vec{k}\cdot\vec{x}}\;, \qquad \Pi_k(t) = \frac{-i\omega_k}{\sqrt{2\omega_k}}\Big(a_k(t) - a^\dagger_{-k}(t)\Big) \tag{4.3}$$

where Ω is the spatial volume and the time evolution of the creation and annihilation operators is through the time evolution operator. ω_k^2 are the same as in (2.13).

The expectation value of particle number operator N_k can be expressed in terms of the fields Φ_k and that of the conjugate momentum Π_k as follows,

$$N_k(t) = \langle a_k^\dagger(t)\; a_k(t)\rangle = \frac{1}{2\;\omega_k}\langle \Pi_k(t)\;\Pi_{-k}(t)\; + \omega_k^2\;\Phi_k(t)\;\Phi_{-k}(t)\;\rangle - \frac{1}{2} \tag{4.4}$$

where the bracket $\langle\cdots\rangle$ means an average over the gaussian density matrix defined by the initial occupation numbers $N_k(t_0)$. The time-dependent distribution (4.4) is interpreted as the quasiparticle distribution function.

As in our study of relaxation in the previous section, we examine separately the cases $T >> T_c$ and $T << T_c$ since the interaction vertices are different (we could study the situation in the most general case, but prefer to study in detail the cases separately to illustrate clearly the different processes).

A. $T >> T_c$

The interaction Hamiltonian in this case is given by

$$H_{int} = \frac{\lambda}{4!}\frac{1}{\Omega}\sum_{\vec{k}1,\vec{k}2,\vec{k}3,\vec{k}4}\Phi_{\vec{k}1}\Phi_{\vec{k}2}\Phi_{\vec{k}3}\Phi_{\vec{k}4}\delta_{\vec{k}1+\vec{k}2+\vec{k}3+\vec{k}4} + \delta m^2(T)\sum_{\vec{k}}\Phi_{\vec{k}}\Phi_{-\vec{k}} \tag{4.5}$$

Taking the derivative of $N_k(t)$ with respect to time and using the Heisenberg field equations, we find

$$\dot{N}_k(t) = \frac{1}{2\,\omega_k}\Big[-\frac{\lambda}{6}\,\langle (\Phi)^3_k(t)\,\Pi_{-k}(t) + \Pi_k(t)\,(\Phi)^3_{-k}(t)\,\rangle \\ - \delta m^2(T)\,\langle \Phi_k(t)\,\Pi_{-k}(t) + \Pi_k(t)\,\Phi_{-k}(t)\rangle\Big] \tag{4.6}$$

where we use the compact notation:

$$(\Phi)^3_k(t) \equiv \frac{1}{\Omega}\sum_{k_1,k_2,k_3}\delta_{k_1+k_2+k_3-k,0}\;\Phi_{k_1}(t)\;\Phi_{k_2}(t)\;\Phi_{k_2}(t) \tag{4.7}$$

In a perturbative expansion care is needed to handle the canonical momentum ($\Pi = \dot{\Phi}$) and the scalar field at the same time because of Schwinger terms. This ambiguity is avoided by defining

$$\langle \Pi_k(t)\;(\Phi)^3_{-k}(t)\rangle = tr\left[\rho(t_0)\;\Pi_k(t)\;(\Phi)^3_{-k}(t)\right] \equiv \lim_{t\to t'}\frac{\partial}{\partial\, t'} tr\left[(\Phi^+)^3_{-k}(t)\;\rho(t_0)\;\Phi^-_k(t')\right] \tag{4.8}$$

Where we used the cyclic property of the trace and the $\pm$ superscripts for the fields refer to field insertions obtained as variational derivatives with respect to sources in the forward time branch ($+$) and backward time branch ($-$) in the non-equilibrium generating functional [33].

We now use the canonical commutation relation between Π and Φ and define the mass counterterm $\delta m^2(T) = (\lambda/6)\Delta(T)$ to write the above expression as

$$\dot{N}_k(t) = -\frac{\lambda}{12\omega_k}\left\{2\frac{\partial}{\partial t'}\left[\langle(\Phi^+(t))^3_k\Phi^-_{-k}(t')\rangle + \Delta(T)\langle\Phi^+_k(t)\Phi^-_{-k}(t')\rangle\right]_{t=t'} \right. \\ \left. +\; 3i\left[\frac{1}{\Omega}\sum_k\langle\Phi^+_k(t)\Phi^-_{-k}(t) + \frac{\Delta(T)}{3}\right]\right\} \tag{4.9}$$

The right hand side of eq. (4.9) can be obtained in weak coupling expansion in λ. Such a perturbative expansion is in terms of the non-equilibrium Green's functions (2.4-2.9) with the basic Green's functions given by (2.11, 2.12) but with the non-equilibrium occupation number $N_k(t_0)$ replacing the equilibrium one n_k. An important point to notice is that these Green's functions include the proper microscopic scales as the contribution of the hard thermal loops have been incorporated by summing the tadpole diagrams. The propagators entering in the calculations are the resummed propagators. The terms with $\Delta(T)$ are required to cancel the tadpoles to all orders.

As will be discussed in detail below, such an expansion will be meaningful for times $t \ll \tau_r = |(N_k(t)/\dot{N}_k(t))|$, where τ_r is the relaxation time scale for the non-equilibrium distribution function. For small enough coupling we expect that τ_r will be large enough so that there is a wide separation between the microscopic and the relaxation time scales

that will warrant such an approximation (see discussion below). At order $\mathcal{O}(\lambda)$ the right hand side of (4.9) vanishes identically. This is a consequence of the fact that the initial density matrix is diagonal in the occupation number. Out of equilibrium, when the *equal time* correlation functions in (4.9) are time dependent, the mass counterterm will acquire a (weak) time dependence through the non-equilibrium distribution functions, thus $\Delta(T,t)$ will be a slowly varying function of time on a scale determined by τ_r. Thus from the formidable expression (4.9) only the first term remains after the hard thermal loop resummation and we find one of our main results:

$$\dot{N}_k(t) = -\frac{\lambda}{6\omega_k}\frac{\partial}{\partial t'}\left[\langle(\Phi^+(t))^3_k \Phi^-_{-k}(t')\rangle\right]_{t=t'} \tag{4.10}$$

with the understanding that no tadpole diagrams contribute to the above equations as they are automatically cancelled by the terms containing $\Delta(T)$ in (4.9).

To two-loop order, the diagrams that contribute to (4.10) are shown in figure 1. We find the following expression for the evolution of the distribution function:

$$\begin{aligned}\dot{N}_k(t) = \frac{\lambda^2}{3}\frac{1}{2\,\omega_k}\int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}\frac{d^3k_3}{(2\pi)^3}\frac{1}{8\omega_{k1}\omega_{k2}\omega_{k3}}(2\pi)^3\delta^3(\vec{k}-\vec{k}_1-\vec{k}_2-\vec{k}_3)\\ \times\Bigg\{\left[\frac{\sin\left[(\omega_k+\omega_{k1}+\omega_{k2}+\omega_{k3})(t-t_0)\right]}{\omega_k+\omega_{k1}+\omega_{k2}+\omega_{k3}}\right]\mathcal{N}_1(t_0)\\ +\left[\frac{3\,\sin\left[(\omega_k+\omega_{k1}+\omega_{k2}-\omega_{k3})(t-t_0)\right]}{\omega_k+\omega_{k1}+\omega_{k2}-\omega_{k3}}\right]\mathcal{N}_2(t_0)\\ +\left[\frac{3\,\sin\left[(\omega_k+\omega_{k1}-\omega_{k2}-\omega_{k3})(t-t_0)\right]}{\omega_k+\omega_{k1}-\omega_{k2}-\omega_{k3}}\right]\mathcal{N}_3(t_0)\\ +\left[\frac{\sin\left[(\omega_k-\omega_{k1}-\omega_{k2}-\omega_{k3})(t-t_0)\right]}{\omega_k-\omega_{k1}-\omega_{k2}-\omega_{k3}}\right]\mathcal{N}_4(t_0)\Bigg\}\end{aligned} \tag{4.11}$$

where

$$\begin{aligned}\mathcal{N}_1(t) &= (1+N_k(t))\,(1+N_{k1}(t))\,(1+N_{k2}(t))\,(1+N_{k3}(t)) - N_k(t)\,N_{k1}(t)\,N_{k2}(t)\,N_{k3}(t)\\ \mathcal{N}_2(t) &= (1+N_k(t))\,(1+N_{k1}(t))\,(1+N_{k2}(t))\,N_{k3}(t) - N_k(t)\,N_{k1}(t)\,N_{k2}(t)\,(1+N_{k3}(t))\\ \mathcal{N}_3(t) &= (1+N_k(t))\,(1+N_{k1}(t))\,N_{k2}(t)\,N_{k3}(t) - N_k(t)\,N_{k1}(t)\,(1+N_{k2}(t))\,(1+N_{k3}(t))\\ \mathcal{N}_4(t) &= (1+N_k(t))\,N_{k1}(t)\,N_{k2}(t)\,N_{k3}(t) - N_k(t)\,(1+N_{k1}(t))\,(1+N_{k2}(t))\,(1+N_{k3}(t))\end{aligned} \tag{4.12}$$

There are several noteworthy features of this expression:

i) $\dot{N}_k(t)$ vanishes at $t=t_0$. This is a consequence of the choice of the initial density matrix which is diagonal in the $N_k(t_0)$ basis, so that the occupation number operator commutes with the density matrix at $t=t_0$.

ii:) consider times much larger than the time scales implicit in the arguments of the sine functions, typically $t-t_0 \gg M_R^{-1}(T)$. Assuming that the rate of change of the occupation numbers is very slow so that during this time the occupation numbers have not changed significantly, namely, $\tau_r \approx N_k(t_0)/\dot{N}(t_0) \gg (t-t_0) \gg M_R^{-1}(T)$, we can approximate

$$\frac{\sin(W(t-t_0))}{W} \approx \pi\delta(W) \tag{4.13}$$

where W is any of the combinations entering in the arguments of the sine functions above. This approximation is the same as that invoked in time-dependent perturbation theory leading to Fermi's Golden Rule. Under the assumptions leading to the approximation (4.13) we recognize that each term in the expression (4.11) can be identified with a term having the same structure in the imaginary part of the self energy given by eq. (3.21), *including off-shell processes* when the non-equilibrium occupation numbers $N_k(t_0)$ are replaced by n_k. The first term describes the creation of four particles minus the destruction of four particles in the plasma, the second and fourth terms describe the creation of three particles and destruction of one minus destruction of three and creation of one, the third term is the *scattering* of two particles off two particles and is the usual Boltzmann term. If the $N_k(t_0)$ coincide with the equilibrium distribution functions n_k then *all the terms* in eq. (4.11) vanish identically and $\dot{N}_k = 0$, including the off-shell terms as a consequence of the energy conservation delta functions.

iii) Using the approximation (4.13) and assuming that all the modes but the one with wavevector k are in equilibrium, and the mode with wavevector k is slightly off-equilibrium, that is $N_{ki} = n_{ki}\,; N_k = n_k + \delta n_k(t_0)$ to linear order in $\delta n_k(t_0)$, we obtain to linear order in the $\delta n_k(t_0)$ the "relaxation time approximation"

$$\frac{d\delta n_k}{dt} = -\frac{\Sigma_I(\omega_k)}{\omega_k}\delta n_k(t_0) \tag{4.14}$$

where $\Sigma_I(\omega_k)$ is given by eq. (3.21). Thus the relation between the rate of change of the non-equilibrium distribution function in the relaxation time approximation and the "damping rate" of quasiparticles becomes explicit.

iv:) Only the third (scattering) term in the evolution equation (4.11) conserves the total number of particles $N = \sum_k N_k$. The other three terms are a consequence of the fact that the interaction *does not* conserve the particle number and this gives rise to off-shell processes that change the particle number. However as explained above, these processes only contribute on time scales comparable to the microscopic scale. These particle-number-changing processes produce departures from chemical equilibrium on short time scales.

Obviously, equations (4.11, 4.14) describe only the early time evolution of the distribution functions. They neglect the change in the initial occupation numbers and, as they stand, cannot be extended to long times. Consider however the following procedure: assume that there is a wide separation of time scales in the sense that $\tau_r \gg M_R^{-1}(T)$, and consider integrating (4.11) (or the linearized version (4.14)) in a time interval from $t = t_0$ to $t_0 + \Delta t$ such that $M_R^{-1}(T) \ll \Delta t \ll \tau_r$. In this time interval we can approximate the sine functions by energy conserving delta functions using (4.13) keeping the occupation numbers at their initial value at $t = t_0$. At the end of this interval "reset" all the occupation numbers to the values $N_k(t_0 + \Delta t)$, and furthermore reduce the density matrix at this time $(t_0 + \Delta t)$ to the diagonal elements in the basis of the occupation numbers at this time. Then the Green's functions will look exactly the same as those given at the initial time $t = t_0$ but in terms of the occupation numbers at $t = t_0 + \Delta t$. In this "coarse grained" density matrix there are no off-diagonal matrix elements of the creation and destruction quasiparticle operators at $t = t_0 + \Delta t$. In particular, this reduction of the density matrix neglects new correlations of the form $\langle a_k(t)a_{-k}(t)\rangle\,; \langle a^\dagger_k(t)a^\dagger_{-k}(t)\rangle$ which will be generated. Now we can iterate the procedure obtaining the equivalent of eq. (4.11) but with the $N_k(t_0 + \Delta t)$ on the right hand side, iterating this procedure for all times is equivalent to the non-linear equation

$$\dot{N}_k(t) = \frac{\lambda^2}{3}\,\frac{1}{2\,\omega_k}\int \frac{d^3k_1}{(2\pi)^3}\,\frac{d^3k_2}{(2\pi)^3}\,\frac{d^3k_3}{(2\pi)^3}\,\frac{1}{8\omega_{k1}\omega_{k2}\omega_{k3}}\,(2\pi)^3\delta^3(\vec{k}-\vec{k}_1-\vec{k}_2-\vec{k}_3)$$
$$\times[\delta(\omega_k+\omega_{k1}+\omega_{k2}+\omega_{k3})\mathcal{N}_1(t)$$
$$+3\,\delta(\omega_k+\omega_{k1}+\omega_{k2}-\omega_{k3})\mathcal{N}_2(t)$$
$$+3\,\delta(\omega_k+\omega_{k1}-\omega_{k2}-\omega_{k3})\mathcal{N}_3(t)$$
$$+\delta(\omega_k-\omega_{k1}-\omega_{k2}-\omega_{k3})\mathcal{N}_4(t)] \tag{4.15}$$

Linearizing the above equation, considering only the mode with wavevector k to be slightly out of equilibrium whereas all the other modes are in equilibrium leads to the "relaxation time approximation"

$$\frac{d\delta n_k}{dt} = -\frac{\Sigma_I(\omega_k)}{\omega_k}\delta n_k(t) \tag{4.16}$$

The only term that gives a non-zero contribution in (4.15) is the third ("scattering") term, because for all the other terms the δ functions have no support on the mass-shell. Therefore we obtain the Boltzmann equation (4.15) and its "relaxation time approximation" (4.16) (where only the second term in (3.21) contributes to $\Sigma_I(\omega_k)$). This result and the interpretation of the damping rate $\Sigma_I(\omega_k)/2\omega_k$ as (half) the relaxation rate of the quasiparticle distribution function was proposed by Weldon [26]. To our knowledge it has not been proven before by obtaining the Boltzmann equation directly from a first principle calculation and analyzing the "coarse graining" approximations that lead to this result.

The linearized approximation gives the time scales for relaxation for situations close to equilibrium. In the case of small momentum or $T >> M_R(T) >> p$ we obtain

$$\tau_r(k\approx 0) \approx \frac{M_R(T)}{\Sigma_I(\omega_{k\approx 0})} \approx \frac{32\sqrt{24\pi}}{\lambda^{3/2}T} \tag{4.17}$$

$$M_R(T)\tau_r \propto \frac{1}{\lambda} \gg 1 \tag{4.18}$$

where we have assumed very weak coupling.

The collisional relaxation rate obtained from the hard thermal loop resummation

$$\Gamma_{coll} = \frac{\lambda^{3/2}T}{32\sqrt{24\pi}} \tag{4.19}$$

is very *different* from the "naive" scattering rate that one would write down for a "classical" Boltzmann equation. Such a collision rate would be obtained as

$$\Gamma_{cla} = \langle n\sigma v\rangle \tag{4.20}$$

where n is the number density, σ the $2\to 2$ scattering cross section, v the mean velocity of the colliding particles and $\langle\cdots\rangle$ the ensemble average. For $T \gg m$ one would obtain $n \approx T^3$, $v \approx 1$, the Born cross section is $\sigma(E) \approx \lambda^2/E^2$ where E is the total energy of the colliding particles, which in the high temperature plasma is $E \approx T$. Such a "classical" estimate would lead to a scattering rate $\Gamma_{cla} \approx \lambda^2 T$ which for very weak coupling grossly underestimates the correct resummed result (4.19) for long wavelength modes.

B. $T < T_c$

In this section we obtain the Boltzmann equation in the broken symmetry case. After a shift of the fields on both branches by the vacuum expectation value as $\Phi^{\pm} \to v + \Phi^{\pm}$, the interaction Hamiltonian contains new terms as compared to the unbroken symmetry case of the previous section. These are given by

$$H^v_{int} = \frac{\lambda v}{6}\left\{\frac{1}{\sqrt{\Omega}}\sum_{\vec{k}1,\vec{k}2,\vec{k}3}\Phi_{\vec{k}1}\Phi_{\vec{k}2}\Phi_{\vec{k}3}\delta_{\vec{k}1+\vec{k}2+\vec{k}3,\vec{0}} + h\sqrt{\Omega}\Phi_{\vec{k}=0}\right\} \tag{4.21}$$

$$h = v^2 - \frac{6m_0^2}{\lambda}\ . \tag{4.22}$$

There is now a one loop contribution to the Boltzmann equation. We will *not* fix the v.e.v. to the tree level value. Rather, h is considered as a source term that will be used to cancel all the tadpoles arising from the cubic interaction, whereas the mass counterterm will cancel the tadpoles corresponding to mass corrections. In this manner, using h to cancel the cubic tadpoles the result for v is the *true* v.e.v. including quantum (and thermal) corrections.

We now proceed in the same manner as in the unbroken symmetry case of the previous section and obtain the evolution equation for the occupation number by using the Heisenberg field equations and the non-equilibrium Feynman rules. Using the canonical commutation relations between the field and its canonical momentum, and requiring that h cancels all the tadpoles arising from the cubic interaction, and $\delta m^2(T)$ cancel the tadpoles corresponding to mass insertions, we find the new contribution to the evolution equation

$$\dot{N}^v_k(t) = -\frac{\lambda v}{2\omega_k}\frac{\partial}{\partial t'}\left[\langle(\Phi^+(t))^2_k\Phi^-_{-k}(t')\rangle\right]_{t=t'} \tag{4.23}$$

$$(\Phi^+(t))^2_k = \frac{1}{\sqrt{\Omega}}\sum_{\vec{k}1,\vec{k}2}\Phi_{\vec{k}1}\Phi_{\vec{k}2}\delta_{\vec{k}1+\vec{k}2\,,\vec{k}} \tag{4.24}$$

This contribution must be added to the obtained in the previous section and given by equation (4.10). To one loop order, the contributions are shown in figure 2 and to this order we obtain the following evolution equation for the distribution functions:

$$\begin{aligned}\dot{N}^v_k(t) = \frac{\lambda^2 v^2}{2\,\omega_k}\int\frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}\frac{1}{4\omega_{k1}\omega_{k2}}(2\pi)^3\delta^3(\vec{k}-\vec{k}_1-\vec{k}_2)\\ \times\Bigg\{\left[\frac{\sin\left[(\omega_k+\omega_{k1}+\omega_{k2})(t-t_0)\right]}{\omega_k+\omega_{k1}+\omega_{k2}}\right]\mathcal{N}^v_1(t_0)\\ +\left[\frac{\sin\ \left[(\omega_k-\omega_{k1}-\omega_{k2})(t-t_0)\right]}{\omega_k-\omega_{k1}-\omega_{k2}}\right]\mathcal{N}^v_2(t_0)\\ +\left[\frac{\sin\left[(\omega_k+\omega_{k1}-\omega_{k2})(t-t_0)\right]}{\omega_k+\omega_{k1}-\omega_{k2}}\right]\mathcal{N}^v_3(t_0)\\ +\left[\frac{\sin\ \left[(\omega_k-\omega_{k1}+\omega_{k2})(t-t_0)\right]}{\omega_k-\omega_{k1}+\omega_{k2}}\right]\mathcal{N}^v_4(t_0)\Bigg\}\ ,\end{aligned} \tag{4.25}$$

where

$$\mathcal{N}_1^v(t) = (1+N_k(t))\,(1+N_{k1}(t))\,(1+N_{k2}(t)) - N_k(t)\,N_{k1}(t)\,N_{k2}(t)$$
$$\mathcal{N}_2^v(t) = (1+N_k(t))\,N_{k1}(t)\,N_{k2}(t) - N_k(t)\,(1+N_{k1}(t))\,(1+N_{k2}(t))$$
$$\mathcal{N}_3^v(t) = (1+N_k(t))\,(1+N_{k1}(t))\,N_{k2}(t) - N_k(t)\,N_{k1}(t)\,(1+N_{k2}(t))$$
$$\mathcal{N}_4^v(t) = (1+N_k(t))N_{k1}(t)\,(1+N_{k2}(t)) - N_k(t)\,(1+N_{k1}(t))\,N_{k2}(t) \tag{4.26}$$

Again under the assumption of a separation of time scales and approximating the sine functions by energy conserving delta functions using (4.13) as before, and considering that all the modes but the mode with wave-vector k are in equilibrium with distributions $N_{ki}(t_0) = n_{ki}$ and $N_k(t_0) = n_k + \delta n_k(t_0)$ we are led to the relaxation time approximation

$$\frac{d\delta n_k}{dt} = -\frac{\Sigma_I^v(\omega_k)}{\omega_k}\delta n_k(t_0) \tag{4.27}$$

with $\Sigma_I^v(\omega_k)$ given by the imaginary part of the one loop contribution to the self energy, equations (3.39) and (3.40). In the broken symmetry case with only one scalar field, the imaginary part $\Sigma_I^v(\omega_k)$ vanishes on-shell because of kinematics. However, it is non-zero in the case in which the particle Φ can decay into lighter scalars χ as discussed in section III-B and below.

The discussion of the previous case also applies here, in that the above evolution equations neglect the change of occupations numbers on the right hand side of the equations, and are thus valid only for early times. However, as discussed in the previous case, if there is a separation of time scales such that $\tau_r M_R(T) >> 1$ we can integrate the evolution equations for times $M_R^{-1}(T) << (t-t_0) << \tau_r$, approximate the sine functions by energy conserving delta functions, and keep the occupation numbers fixed at their values at t_0. Furthermore reducing the density matrix to the diagonal elements in the basis of occupation numbers at time t, we can iterate this equation leading to the usual Boltzmann equation. The new terms describe the evolution of the distribution function arising from processes of decay and recombination in the plasma. Such an iterative procedure leads to the following one-loop contribution to the Boltzmann equation

$$\begin{aligned}\dot{N}_k^v(t) = \frac{\lambda^2 v^2}{2\,\omega_k}\int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}\frac{1}{4\omega_{k1}\omega_{k2}}(2\pi)^3\delta^3(\vec{k}-\vec{k}_1-\vec{k}_2)\\ \times\delta(\omega_k+\omega_{k1}+\omega_{k2})\mathcal{N}_1^v(t)\\ +\delta(\omega_k-\omega_{k1}-\omega_{k2})\mathcal{N}_2^v(t)\\ +\delta(\omega_k+\omega_{k1}-\omega_{k2})\mathcal{N}_3^v(t)\\ +\delta(\omega_k-\omega_{k1}+\omega_{k2})\mathcal{N}_4^v(t)\ .\end{aligned} \tag{4.28}$$

This one-loop result also applies if the scalar field Φ couples trilinearly to lighter scalars χ, namely with a vertex $g\Phi\chi^2$ and we assume that kinematical decay is possible, i.e. $M_R(T) > 2m_\chi(T)$. Under the assumption of separation of time scales, which is the case for small coupling, justifying the replacement of the sine by delta functions we obtain a Boltzmann equation for evolution of the distribution function for the Φ field which has the same form as (4.28), but with $\lambda^2 v^2/2$ replaced by $2g^2$ and the functions $\mathcal{N}_i^v(t)$ replaced with

$$\mathcal{N}_1^\chi(t) = (1+N_k(t))\,(1+N_{k1}^\chi(t))\,(1+N_{k2}^\chi(t)) - N_k(t)\,N_{k1}^\chi(t)\,N_{k2}^\chi(t)$$
$$\mathcal{N}_2^\chi(t) = (1+N_k(t))\,N_{k1}^\chi(t)\,N_{k2}^\chi(t) - N_k(t)\,(1+N_{k1}^\chi(t))\,(1+N_{k2}^\chi(t))$$

$$\mathcal{N}_3^\chi(t) = (1+N_k(t))\,(1+N_{k1}^\chi(t))\,N_{k2}^\chi(t) - N_k(t)\,N_{k1}^\chi(t)\,(1+N_{k2}^\chi(t))$$
$$\mathcal{N}_4^\chi(t) = (1+N_k(t))N_{k1}^\chi(t)\,(1+N_{k2}^\chi(t)) - N_k(t)\,(1+N_{k1}^\chi(t))\,N_{k2}^\chi(t)\ . \tag{4.29}$$

Here $N_k(t)$ and $N_k^\chi(t)$ are the non-equilibrium distribution functions for the quanta of the fields Φ and χ respectively and $\omega_{k1,2} = \sqrt{k_{1,2}^2 + m_\chi^2(T)}$. The term proportional to $\mathcal{N}_2^\chi(t)$ describes the decay process of $\Phi \to \chi\chi$ minus the "recombination" $\chi\chi \to \Phi$ in the medium. The relaxation time approximation for the evolution of $N_k(t) = n_k + \delta n_k(t)$ while the N_k^χ are the equilibrium functions is given by

$$\frac{d\delta n_k}{dt} = -2\Gamma_{\Phi\leftrightarrow\chi\chi}\delta n_k(t) \tag{4.30}$$

with $\Gamma_{\Phi\leftrightarrow\chi\chi}$ the decay minus recombination rate given by equation (3.42).

Thus, after this analysis the nature and validity of the approximations and resummations implied in the usual Boltzmann equation (and its linearized version) become clear:

i) For high temperatures a hard thermal loop resummation to incorporate the proper microscopic time scales. Such a resummation is needed to incorporate the relevant time for "coarse graining".

ii) A wide separation between the microscopic time scale and the relaxation time which, as shown by the estimate (4.18), is warranted for very weak coupling. Such a case corresponds to the collisional and decay lifetime of the particle being much larger than the microscopic time scale given by the inverse mass of the particle in the medium, namely a "sharp resonance" condition. This separation of time scales permits the identification of intermediate times $M_R^{-1}(T) \ll t \ll \tau_r$ such that the "Fermi Golden Rule" approximation (4.13) is justified and energy conservation is enforced.

iii) A reduction of the density matrix to the diagonal elements in terms of the time dependent occupation number. This reduction neglects higher order correlations of the quasiparticle operators such as $\langle a_k(t)a_k(t)\rangle$ and $\langle a_k^\dagger(t)a_k^\dagger(t)\rangle$. Therefore the "coarse graining" procedure is not only at the level of spatial and temporal averages but also of reducing the density matrix, thus reducing the hierarchy of equations of motion.

Although all of these approximations are implicit in the usual procedure through the Wigner transform and the gradient expansion, our method illuminates all the necessary steps and the physical justifications at once.

C. Improvements:

This method allows us to identify several improvements to the description the kinetics of thermalization and relaxation:

i) The first improvement is to incorporate wave-function renormalization in the definition of the quasiparticles in the Hamiltonian as well as coupling constant renormalization. For this we write the renormalized fields, canonical momentum and coupling constant as

$$\Phi_R = Z_\Phi^{-1/2}\Phi\,;\ \ \Pi_R = Z_\Phi^{1/2}\Pi\,;\ \ \lambda_R = Z_\Phi^2 Z_\lambda^{-1}\lambda \tag{4.31}$$

Notice that Π_R and Φ_R are canonical conjugate variables. The "free quasiparticle" Hamiltonian is now written in terms of Π_R, Φ_R and $M_R(T)$ and the difference between the bare

and renormalized quantities is accounted for in the interaction part of the Hamiltonian. The quasiparticle distribution $N_k(t)$ is defined as in eq. (4.4) but in terms of the renormalized fields and canonical momenta. Its evolution equation is obtained from the Heisenberg equations of motion with the new terms in the interaction Hamiltonian as described previously.

There will now be new terms in the equations for the evolution of the distribution function that will cancel the (divergent) contributions arising from the wave function and coupling constant renormalization. These terms will appear at three loops (for coupling constant) and beyond but in terms of the quasiparticle propagators. This is precisely the BPHZ program of renormalization but here implemented in a non-equilibrium formulation. Clearly such an improvement comes at the cost of a complicated structure but this method allows us to incorporate such corrections in a systematic manner.

ii) Rather than assuming a wide separation of time scales and using the approximation (4.13) leading to energy conserving delta functions, one can integrate the evolution equations (4.11, 4.25) numerically in increments $t = t_0 + \Delta t$, "reseting" the values of the occupation numbers for the next increment to those obtained in the previous iteration. Such an approximation can be implemented numerically just as in the usual case. However the equation for the evolution of the distribution functions will now have *off-shell* contributions that will modify the evolution on time scales comparable to $M_R^{-1}(T)$. These contributions include information on the preparation of the non-equilibrium state and make the derivative of the occupation number vanish at $t = t_0$ as it must (the same feature appears in the derivation of Fermi's Golden Rule).

In the two-loop collisional contribution, the energy-conserving delta functions introduce one constraint. Thus a five dimensional k-integral remains, whereas in the "improved" form, without energy conservation one must face the full six-dimensional k-integral. Furthermore, this "improved" set of evolution equations include terms that change chemical equilibrium (do not conserve particle number) on microscopic time scales. These result from the fact that the interactions considered in this study do not conserve particle number and may lead to new effects in the quasiparticle distribution at long times. Such a possibility should be studied further.

V. CONCLUSIONS AND IMPLICATIONS

In this article we have presented a new approach, based on non-equilibrium quantum field theory, to study linear relaxation of quasiparticles and the kinetic description of the evolution of the quasiparticle distribution functions in a scalar field theory. Our method incorporates the relevant microscopic time scales through the resummation of hard thermal loops in the description of both relaxation and kinetics. The method leads to a systematic and consistent scheme for obtaining both the linearized equations of motion and the equations for the distribution functions. It clearly displays the different "coarse graining" approximations usually involved in a kinetic description and their range of validity. It allows for a systematic and consistent improvement in the kinetic description, including off-shell and renormalization effects, as well as processes that change chemical equilibrium on short time scales, and is numerically implementable. These improvements in the kinetic equations may prove physically relevant if the time scale for relaxational and thermalization processes

is not widely different from the microscopic time scales and off-shell processes must be taken into account for a more accurate description.

As a result of our study of both relaxation and kinetics we established a direct proof of the relationship first proposed by Weldon [26] between the damping rate of the quasiparticles and the relaxation rate of the quasiparticle distribution function in the relaxation time approximation after the coarse graining of the kinetic equations.

We have studied relaxation and kinetics both in the unbroken symmetry phase for $T >> T_c$ as well as in the ordered, broken symmetry phase for $T < T_c$. The hard thermal loop resummation is very important in the unbroken phase to obtain the proper microscopic time scales and to establish a wide separation of time scales that justifies a kinetic description in the weak coupling regime. The broken symmetry case provides a setting to incorporate processes in which the scalar particle can decay, via a trilinear coupling, to other lighter scalars in the theory. We studied the effect of such processes both on relaxation and kinetics, obtaining the kinetic equations that include decay and recombination in the medium. We also analyzed the effect of processes that only contribute below the light cone that are similar to Landau damping. We pointed out that although these processes do not contribute *directly* to relaxation and damping rates, they contribute *indirectly* through the sum rules obeyed by the spectral function.

The methods developed here are suitable for treating the evolution of the distribution functions for the particles produced during the parametric amplification stage in inflationary cosmological scenarios [21–24] and in the supercooling stage of the quark-gluon transition [25]. The reason is that these methods can be simply extended to include the time evolution of the expectation value of the scalar fields (mean fields) into the kinetic equations. If there is a separation of time scales between the stage of particle production and the onset of thermalization, these equations will lead to a description of particle production via parametric amplification *and* the collisional processes leading to thermalization and completion of the reheating stage. We are currently applying these techniques to that problem, and we expect that these methods will be extended to gauge and fermion theories in the near future.

Some questions that this study poses that are important are the following: is there a kinetic description of "critical slowing down" near a phase transition? can a kinetic description be extended to temperatures near criticality when long range correlations begin to form and relaxational time scales should diverge?

These questions are relevant for a deeper understanding of the dynamics of phase transitions and the non-equilibrium description of thermalization and relaxation of long-wavelength fluctuations. We expect to address these and other related questions in the near future.

ACKNOWLEDGMENTS

D.-S. Lee would like to thank J. Ng, B.-L. Hu and L. Dolan for interesting discussions. D.B. would like to thank H. J. de Vega, M. D'Attanasio, R. Holman, R. Pisarski, H. A. Weldon, P. Elmfors, P. Henning and J. L. Goity for illuminating conversations during the early stages of this work. He also thanks the N.S.F. for partial funding through grant award: PHY-9302534. D.-S. Lee was supported in part by DOE grant DE-FG05-85ER-40219-TASKA. I.D.L. thanks the U.K. Particle Physics and Astronomy Research Council

for a travel grant.

REFERENCES

[1] For reviews on the subject see: B. Mueller in *Particle Production in Highly Excited Matter*, p. 11, H. H. Gutbrod and J. Rafelski Editors, Plenum Press, N.Y., 1993; B. Mueller, Rep. of Prog. in Phys. 58, 611 (1995).

[2] P. Danielewicz, Ann. Phys. (N.Y.) 152, 239 (1984). S. Mrowczynski and P. Danielewicz, Nucl. Phys. B342, 345 (1990).

[3] H.T. Elze and U. Heinz, Phys. Rep. 183, 81(1989) and references therein;

[4] E. Calzetta and B.L. Hu, Phys. Rev. D37, 2787(1988).

[5] P. Henning, Phys. Rep. 253, 235 (1995) and references therein.; P. Henning, *Quantum Transport Theory: Beyond Molecular Dynamics* presented at Int. Workshop on Finite Temperature Field Theory, Dalian, China, Aug. 7-11, 1995 (to be published in the proceedings, Ed. by V. Khanna and Y.Gui, World Scientific, 1996), hep-ph/9510315.

[6] J. Rau and B. Mueller, *From Reversible Quantum Microdynamics to Irreversible Quantum Transport*, nucl-th/9505009.

[7] P. Zhung and U. Heinz, *Relativistic Kinetic Equations for Electromagnetic, Scalar and Pseudoscalar Interations*, hep-ph/9510206.

[8] K. Geiger, *Quantum Field Kinetics of QCD: Quark-Gluon Transport for Ligh-cone Dominated Processes*, hep-ph/9507365.

[9] V. Klimov, Sov. J. Nucl. Phys. 33, 934 (1981); H. A. Weldon, Phys. Rev. D26, 2789 (1982).

[10] For a clear discussion see: J. P. Blaizot, J-Y. Ollitrault and E. Iancu, *Collective Phenomena in the Quark Gluon Plasma* in *Quark-Gluon Plasma II* Ed. by R. C. Hwa (World Scientific, Singapore, 1995); J. P. Blaizot and E. Iancu, *Non-perturbative aspects of screening phenomena in abelian and non-abelian gauge theories* Saclay preprint T-95/055 (1995). See also [16].

[11] S. Jeon, *Hydrodynamic Transport Coefficients*, (to appear in Phys. Rev. D 1996); S. Jeon and L. Yaffe, *From Quantum Field Theory to Hydrodynamics: Transport Coefficients and Effective Kinetic Theory*, hep-ph/9512263.

[12] W. M. Zhang and L. Wilets, Phys. Rev. C45, 1900 (1992) and references therein.

[13] S. Mrowczynski and U. Heinz, Ann. Phys. (N.Y.) 229, 1 (1994).

[14] J. Hufner, S. P. Klebansky, P. Zhuang and H. Voss, Ann. Phys. (N.Y.) 234, 225 (1994); P. Zhuang, J. Hufner and S. P. Klebansky, Nucl. Phys. A 576, 525 (1994).

[15] E. Braaten and R. D. Pisarski, Nucl. Phys. B337, 569(1990);ibid B339, 310(1990); E. Braaten and R. D. Pisarski, Nucl. Phys. B337, 569(1990);ibid B339, 310(1990); R. D. Pisarski, Phys. Rev.Lett. 63,1129(1989); R. D. Pisarski, Phys. Rev.Lett. 63,1129(1989).

[16] J.-P. Blaizot in *Proceedings of Fourth Summer School and Symposium on Nuclear Physics* Eds. D.P. Min and M. Rho (1991).

[17] I. D Lawrie, J. Phys. A21, L823 (1988); A25 6493 (1992); Phys. Rev. D30 3330 (1989); in *Workshop on Perturbative Methods in Hot Gauge Theories*, Winnipeg -July 20-25, 1992: Published in Can. J. Phys. 71, 262 (1993).

[18] For early reviews on the inflationary scenario see for example: R. H. Brandenberger, Rev. of Mod. Phys. 57, 1 (1985); Int. J. Mod. Phys. A2, 77 (1987); L. Abbott and S. Y. Pi, *Inflationary Cosmology*, World Scientific, 1986; A. D. Linde, Rep. Prog. Phys. 47, 925 (1984).

[19] E. W. Kolb and M. S. Turner, *The Early Universe*, Addison Wesley (Frontiers in Physics) (1990).
[20] A. D. Linde, *Particle Physics and Inflationary Cosmology*, Harwood, Chur, Switzerland, 1990; *Lectures on Inflationary Cosmology*, in *Current Topics in Astrofundamental Physics*, *The Early Universe*, Proceedings of the Chalonge Erice School, N. Sánchez and A. Zichichi Editors, Nato ASI series C, vol. 467, 1995, Kluwer Acad. Publ.; *Quantum Cosmology and the Structure of Inflationary Universe*, gr-qc/9508019 (1995).
[21] L. Kofman, A. D. Linde, and A. A. Starobinsky, Phys. Rev. Lett. 73, 3195 (1994) and *Non-Thermal Phase Transitions After Inflation*, hep-th-9510119 (1995).
[22] Y. Shtanov, J. Traschen and R. Brandenberger, Phys. Rev. D51, 5438 (1995).
[23] D. Boyanovsky, H. J. de Vega, R. Holman, D.-S. Lee and A. Singh, Phys. Rev. D51, 4419 (1995); D. Boyanovsky, M. D'Attanasio, H.J. de Vega, R. Holman and D,-S. Lee, *Linear vs. Non-Linear Relaxation: Consequences for Reheating and Thermalization* (to appear in Phys. Rev. D) hep-ph/9507414.
[24] M. Yoshimura, *Catastrophic particle production under periodic perturbation*, hep-th-9506176; *Particle production in dissipative cosmic field*, hep-ph-9508378 (1995); H.Fujisaki, K. Kumekawa, M. Yamaguchi and M. Yoshimura,*Particle Production and Gravitino Abundance after Inflatio*, hep-ph/9511381.
[25] S. Mrowczynski and B. Muller, Phys. Lett. B. 363, 1 (1995).
[26] H. A. Weldon, Phys. Rev. D28, 2007 (1983); Ann. of Phys. (N.Y.) 228, 43 (1993).
[27] R. R. Parwani, Phys. Rev. D45, 4695(1992).
[28] J. Schwinger, J. Math. Phys. 2, 407 (1961).
[29] L.V. Keldysh, JETP 20, 1018 (1965).
[30] K. T. Mahanthappa, Phys. Rev. 126, 329 (1962); P. M. Bakshi and K.T. Mahanthappa, J. Math. Phys. 4 1,12 (1963).
[31] for a review and further references see: K. Chou, Z. Su, B. Hao and L. Yu, Phys. Rep. 118, 1 (1985); A. Niemi and G. Semenoff, Ann. of Phys. (NY) 152, 105 (1984); N.P. Landsmann and C.G. van Weert, Phys. Rep. 145, 141 (1987); E. Calzetta and B.L. Hu, Phys. Rev. D 41, 495 (1990); *ibid* 37, 2838 (1990); J.P. Paz, Phys. Rev. D 41, 1054 (1990), *ibid* D 42, 529 (1990).
[32] J. Kapusta, *Finite Temperature Field Theory*, Cambridge Univ. Press. (1989).
[33] D. Boyanovsky, M. D'Attanasio, H.J. de Vega, R. Holman and D.-S. Lee, *Reheating and thermalization: Linear versus nonlinear relaxation*, hep-ph/9507414, to appear in Phys. Rev. D (1996). For a review see: D. Boyanovsky, H. J. de Vega and R. Holman, in the Proceedings of the Second Paris Cosmology Colloquium, Observatoire de Paris, June 1994, p. 127-215, H. J. de Vega and N. Sánchez Editors, World Scientific, 1995; D. Boyanovsky, H. J. de Vega and R. Holman, in *Advances in Astrofundamental Physics*, Erice Chalonge Course, N. Sánchez and A. Zichichi Editors, World Scientific, 1995.
[34] Da-Shin Lee, Ph.D. Thesis (University of Pittsburgh, 1995).
[35] P. Elmfors, K. Enqvist and I. Vilja, Nucl. Phys. B422[FS], 521(1994).
[36] L. Dolan and R. Jackiw, Phys. Rev. D9, 3320 (1974).
[37] E. Wang and U. Heinz, *The Plasmon in Hot ϕ^4 Theory*, hep-ph/9509333; E. Wang, U. Heinz and X. Zhang, *Spectral functions for composite fields and viscosity in hot scalar field theory*, hep-ph/9509334.
[38] R. D. Pisarski, Physica A158, 146 (1989).

[39] D. Boyanovsky, M. D'Attanasio, H. J. de Vega and R. Holman, *Evolution of Inhomogeneous Condensates After Phase Transitions*, hep-ph/9602232.

Figure Captions:

Fig. 1 Top: two-loop diagrams contributing to the equation of motion, the dashed line represents an insertion of the non-equilibrium expectation value. Bottom: two-loop diagrams contributing to the Boltzmann equation.

Fig. 2 Top: one-loop diagrams contributing to the equation of motion in the broken symmetry phase. Tadpoles have been absorbed in a mass and v.e.v. renormalization. The dashed line represents an insertion of the non-equilibrium expectation value. Bottom: one loop diagrams contributing to the Boltzmann equation in the broken symmetry phase.